Solid state physics 2021 Minitest 1 (120 minutes)

23 February 2021

These are example solutions for your reference under the following conditions.

- If these solutions contain mistakes (and they may), the physical correctness has priority over them in grading.
- You may not distribute or repost this document.

1. (50 points) **Anisotropy in a three-dimensional bosonic dispersion**

   We consider bosons in three dimensions with an anisotropic dispersion relation $\omega = \gamma_1(k_x^2 + k_y^2) + \gamma_2 k_z^2$, ($\gamma_1 \neq \gamma_2$) and with only one polarization. (Such particles exist and they are magnetic excitations in some materials—magnons).

   (a) (5 points) Explain in words the concepts of dispersion relation and density of states.

   Solution: *Dispersion relation:* The relation between the energy (or equivalently, frequency) of a mode and its wavevector. *Density of states:* The number of modes(/states) per unit energy.

   (b) (15 points) Compute the density of states of these bosons. Check that your resulting $g(\omega)$ scales with $\sqrt{\omega}$ (if you are unable to solve this question, use $g(\omega) = C\sqrt{\omega}$ in the next subquestions). *Hint: to find the density of states, one needs to restore the spherical symmetry in the integral with a wavevector substitution.*

   Solution: We want to find the density of states $g(\omega)$, which relates to reciprocal space by

$$\sum_k \to \frac{L^3}{(2\pi)^3}\int d\vec{k} = \int g(\omega)d\omega. \tag{1}$$

   We substitute $\kappa_x = \sqrt{\gamma_1}k$, $\kappa_y = \sqrt{\gamma_1}k_y$, $\kappa_x = \sqrt{\gamma_2}k_z$, s.t.

$$\omega = \gamma_1(k_x^2 + k_y^2) + \gamma_2 k_z^2 = \kappa^2. \tag{2}$$

   This restores radial symmetry in integration and permits us to integrate over a spherical shell in κ-space,

$$\frac{L^3}{(2\pi)^3}\int d\vec{k} = \frac{L^3}{(2\pi)^3}\frac{1}{\gamma_1\sqrt{\gamma_2}}\int 4\pi\kappa^2 d\kappa. \tag{3}$$

   Next, we express as a function of ω via

$$\kappa^2 d\kappa = \frac{1}{2}\sqrt{\omega}d\omega, \tag{4}$$

and obtain the density of states

$$g(\omega) = \frac{L^3}{(2\pi)^2} \frac{\sqrt{\omega}}{\gamma_1 \sqrt{\gamma_2}}. \tag{5}$$

(c) (10 points) Compute the heat capacity in the low temperature limit. Leave the definite integral unevaluated as long as it does not depend on any parameters like γ or β.

Solution: For the bosonic normal modes, the expression for energy is given by

$$E = \int_0^{\omega_D} g(\omega)\hbar\omega \left[n(\omega, T) + \frac{1}{2}\right] d\omega. \tag{6}$$

Next, fill in density of states and compile integrals,

$$E = \hbar C \int_0^{\omega_D} \frac{\omega^{3/2}}{e^{\beta\hbar\omega - 1}} d\omega + \text{T independent part} \tag{7}$$

Substitute $x = \beta\hbar\omega$, and find

$$E = \frac{\hbar C}{(\beta\hbar)^{5/2}} \int_0^{x_D} \frac{x^{3/2}}{e^x - 1} dx, \tag{8}$$

where the integral converges to some constant D for low T, due to the upper bound going to ∞. The heat capacity becomes

$$\mathcal{C} = \frac{dE}{dT} = \frac{5CD}{2\hbar^{3/2}} T^{3/2}. \tag{9}$$

(d) (10 points) Assuming there is a total number N of these bosonic modes, compute the Debye cutoff frequency ω_D.

Solution: Total number of states N is equal to integrating the density of states up to the cutoff frequency ω_D, s.t.

$$N = C \int_0^{\omega_D} \sqrt{\omega} d\omega. \tag{10}$$

Evaluating the integral results in

$$\omega_D = \left(\frac{3N}{2C}\right)^{2/3} = \left(\frac{6\pi^2 N \gamma_1 \sqrt{\gamma_2}}{L^3}\right)^{2/3}. \tag{11}$$

(e) (10 points) Determine the heat capacity in the high temperature limit. What temperature can be considered high in this case?

Solution:

$$\mathcal{C} = N k_B, \tag{12}$$

which can be found either by stating Dulong-Petit, or Taylor expansion of the integral via

$$E = \frac{\hbar C}{(\beta\hbar)^{5/2}} \int_0^{x_D} \frac{x^{3/2}}{e^x - 1} dx = \frac{2}{3} \frac{\hbar C}{(\beta\hbar)^{5/2}} (\beta\hbar\omega_D)^{3/2} = N k_B T. \tag{13}$$

Temperature T is high when large compared to T_{Debye}.

2. (40 points) **Electron and hole-conductivity in the Drude model**

We consider a semiconductor, which is a material that can host two types of charge carriers. On the one hand there are electrons with charge $-e$ and mass m_e. On the other hand there are holes: particles with charge $+e$ and mass m_h. The concentrations of the electrons and holes are both equal to n.

(a) (10 points) Write down the equations of motion (one for the electrons and one for the holes) describing the average acceleration of the electrons and holes in an electric field E. Include a damping force as in the Drude model.

Solution:

$$m_e \frac{d\langle \vec{v_e} \rangle}{dt} = -e\vec{E} - \frac{m_e \langle \vec{v_e} \rangle}{\tau}, \tag{14}$$

$$m_h \frac{d\langle \vec{v_h} \rangle}{dt} = e\vec{E} - \frac{m_h \langle \vec{v_h} \rangle}{\tau}. \tag{15}$$

(b) (10 points) From the equations of motion, derive an expression for the total electrical conductivity $\sigma = 1/\rho$ as a function of n, m_e, and m_h.

Solution: Steady state problem:

$$0 = -\frac{eE}{m_e} - \frac{\langle \vec{v_e} \rangle}{\tau}, \tag{16}$$

$$0 = \frac{eE}{m_h} - \frac{\langle \vec{v_h} \rangle}{\tau}, \tag{17}$$

$$\vec{v_e} = -\frac{e\tau}{m_e} E, \tag{18}$$

$$\vec{v_h} = \frac{e\tau}{m_h} E, \tag{19}$$

$$j_e = -ne\vec{v_e} = \frac{ne^2\tau}{m_e} E, \tag{20}$$

$$j_h = ne\vec{v_h} = \frac{ne^2\tau}{m_h} E, \tag{21}$$

$$j_{tot} = j_e + j_h = ne^2\tau \left(\frac{1}{m_e} + \frac{1}{m_h} \right) E, \tag{22}$$

$$\sigma_{tot} = ne^2\tau \left(\frac{1}{m_e} + \frac{1}{m_h} \right). \tag{23}$$

(c) (10 points) We will now derive how we expect the conductivity of this semiconductor to scale with temperature. To do so, we first analyze the effect of temperature on the scattering time τ.

Assuming that the scattering rate $1/\tau$ is linearly proportional to the number of thermally excited phonons in the system, argue how you expect τ to scale with temperature T when $T \gg T_D$. (Hint: recollect how the number of phonons occupying one mode behaves at high temperatures.)

Solution: At $T >> T_D$, the number of phonons in a single mode scales proportional to T as follows

$$n_B = \frac{1}{e^{\frac{\hbar\omega}{k_B T}} - 1} \approx \frac{k_B T}{\hbar\omega}, \tag{24}$$

This means that the scattering time τ scales proportional to $\frac{1}{T}$. Therefore,

$$\sigma = \frac{ne^2\tau}{m} \propto \frac{1}{T}. \tag{25}$$

(d) (10 points) The concentrations of free electrons and holes in a semiconductor are not constant like in metals, but rather depend on temperature T: $n \propto e^{-E_g/k_B T}$, where $E_g \gg k_B T$ is a constant. How does the total conductivity of this semiconductor scale with temperature? Argue whether the change in τ or that in n dominates.

Solution:

$$\sigma_{tot} = ne^2\tau\left(\frac{1}{m_e} + \frac{1}{m_h}\right) \tag{26}$$

$$\sigma_{tot} \propto e^{\frac{-E_g}{k_B T}}\frac{1}{T} \tag{27}$$

Since n changes faster than τ due to exponential dependence on T, n dominates the conductivity.